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Rapid Note

Affine turbulence

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Abstract. We present a generalization of the multiplicative model for velocity increments involving an affine process. The consequences on the shape of the probability distribution functions for the velocity increments are explored, and shown to be better compatible with the existence of a scale variation of the skewness.

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1 Introduction

Turbulent flows generally involve statistical variations over a wide range of scales. Because of the scale invariance of the Navier-Stokes equations in the limit of large Reynolds number [9], the description of this scale variation is often done *via* a multiplicative process, coupling the different scales. An elegant consequence of this hypothesis was recently formulated by Castaing and collaborators [3–5,8], who described the evolution of the velocity increments distribution in term of a propagator. The probability distribution of the velocity increments δu at scale ℓ is then linked to the probability distribution at another scale ℓ' *via*:

$$P(\delta u,\ell)=\int G_{\ell\ell'}(a)P\left(\frac{\delta u}{a}\right)\frac{\mathrm{d}a}{a}, \tag{1}$$

where $G_{\ell\ell'}$ is the propagator from the scale ℓ to the scale ℓ'. The problem with this formulation is that it "conserves the skewness"- in other words, the integral $\int_0^\infty P(\delta u,\ell)$ remains constant with scale if P obeys (1). This property is not observed in real turbulence data, where the probability distributions changes from a nearly symmetric distribution at large scale towards a skewed distribution at very small scales. This led people to postulate that the formulation (1) is valid only for the symmetrical part of the distribution.

This shows that turbulence is probably not a pure multiplicative process, at least for longitudinal velocity increments[1]. Actually, a recent investigation of Friedrich *et al.* [1,2] shows that the turbulence is better described by a Langevin process involving both a multiplicative noise, and an additive noise. In other words, to go from one velocity increments at one scale to another velocity increments at another scale, one should perform a multiplication and a translation, *i.e.* an affine transformation. The goal of this note is to investigate the possible generalization of the Castaing formulation if the turbulence is generated by an affine process, and to see whether this simple generalization allows for some skewness generation along the scale, as observed in turbulence.

2 Affine generalization of Castaing's formula

2.1 Formulation

Consider two velocity increments at scale ℓ and scale ℓ', and let us assume that there is a statistical affine connection between them, under the form:

$$\delta u_{\ell'}=a_{\ell'\ell}\delta u_\ell-b_{\ell'\ell}, \tag{2}$$

where $a_{\ell'\ell'}\geq 0$ and $b_{\ell'\ell}$ are two random variables which can be correlated. Note that if b is identically zero, we find the usual multiplicative model for velocity increments. If a is identically 1, we find an additive model for the velocity increments, which was proposed *e.g.* in the case of 2D

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[1] Transverse velocity increments distributions are symmetric and could therefore be described by (1).

turbulence by [10]. In terms of the probability distribution function, this hypothesis can also be written:

$$P(\delta u, \ell) = \int\int P\left(\frac{\delta u - b}{a}, \ell'\right) F_{\ell\ell'}(a,b)\frac{\mathrm{d}a}{a}\mathrm{d}b, \qquad (3)$$

where $F_{\ell\ell'}$ is the joint probability of the multicative and additive variable. It is now our propagator to go from scale ℓ to scale ℓ' and is, as a joint probability, uniquely defined. Note that the representation (3) is reminiscent of multi-resolution transformations. The practical inversion problem, that is finding F from experimentally given $P(\delta u, \ell)$ and $P(\delta u, \ell')$ is a non-trivial optimization problem under certain constrains (like the positivity of F, or the value of its integral), which would be worth studying from a mathematical point of view. Leaving this interesting question for further study, we now investigate the properties of this generalized representation.

2.2 Group structure

If the formulation (3) has any physical ground, it should possess a group structure: going from one scale ℓ to scale ℓ' and then from scale ℓ' to scale ℓ'' should be equivalent to going from scale ℓ to scale ℓ''. This requirement imposes a composition law condition over the function F, namely, that:

$$F_{\ell\ell''}(a', b') = \int\int F_{\ell\ell'}(a,b)F_{\ell'\ell''}\left(\frac{a'}{a}, \frac{b'-b}{a}\right)\mathrm{d}a\mathrm{d}b. \quad (4)$$

In the case where F is independent of b (multiplicative case), we find the usual convolution property in log-variable, which has been established by Castaing [5]. Formula (4) can therefore be seen as a sort of stability property by a generalized convolution. In the multiplicative case, it is well known that the stability condition selects one general class of probability distribution, namely the log-infinitely divisible distributions (see *e.g.* [8]). In the present case, we have a generalization of this result, which we call infinite affine divisibility.

2.3 Infinite "affine" divisibility

2.3.1 Matrix formulation

To define this condition of infinite "affine" divisibility, it is convenient to consider the constraint imposed by (3) on the moments of the probability distribution. Multiplying both sides of (3) by $(\delta u)^n$ and taking the integral with respect to δu, we get the following moment relation:

$$\langle(\delta u_\ell)^n\rangle = \sum_{p=0}^{n} C_n^p \int \delta u^p P(\delta u, \ell')\mathrm{d}\delta u \int\int a^p b^{n-p} F_{\ell\ell'}(a,b)\mathrm{d}a\mathrm{d}b, \qquad (5)$$

where C_n^p is the usual combinatorial coefficient appearing in the development of $(a+b)^n$. This relation can be put into an elegant matrix formulation by introducing the "moment vector" at scale ℓ:

$$X_\ell = \begin{pmatrix} 1 \\ \langle\delta u_\ell\rangle \\ ... \\ \frac{1}{n!}\langle(\delta u_\ell)^n\rangle \\ \ldots \end{pmatrix} \qquad (6)$$

and the propagator matrix $A_{\ell\ell'}$, made of components whose expression at the nth line and pth column is:

$$A_{\ell\ell'}^{np} = \frac{1}{(n-p)!}\int\int a^p b^{n-p} F_{\ell\ell'}(a,b)\mathrm{d}a\mathrm{d}b. \qquad (7)$$

The moment relation then takes the very simple matrix shape

$$X_\ell = A_{\ell\ell'} X'_\ell. \qquad (8)$$

Now, we can obtain our infinite divisible property by assuming that for any ℓ and ℓ', we can build a sequence of scales ℓ_0,...,ℓ_N going from $\ell = \ell_0$ to $\ell_N = \ell'$ for which the matrix $A_{\ell_i\ell_{i+1}}$ tends to to a universal matrix A, which is scale independent. The number of steps $N_{\ell\ell'}$ required to go from ℓ to ℓ' in this universal sequence is called the depth of the cascade. The elementary step required to go from one scale of the sequence to the next is called the step of the cascade. In standard log-infinite divisible distributions, the step tends to zero, and $N_{\ell\ell'}$ is proportional to $\ln(\ell/\ell')$; in turbulence, the step appears to be finite and scale dependent (of the order of $\ln(1+\ell_0/\ell)$ where ℓ_0 is of the order of the Kolmogorov scale [11]). As a consequence, N does not vary like in the true log-infinitely divisible case, but rather obeys $N_{\ell\ell'} \sim 1-(\ell/\ell')^\mu$, where μ tends to zero as the Reynolds number increases [3,5,8]. This variation can be predicted from symmetry argument, by taking into account the finiteness of N induced by the finiteness of the step of the cascade [12,13].

In any case, whatever the shape of the function $N_{\ell\ell'}$, we can now simply relate the moment vector at any given scale -say the largest one, L, to the vector moment at scale ℓ *via*:

$$X_\ell = A^{N_{\ell L}} X_L, \qquad (9)$$

where A is the "elementary" matrix. This relation generalizes the log-infinite divisible property because in the case where b, the additive parameter is zero, $F(a,b) = G(a)\delta(b)$, and one can easily check that $A_{\ell\ell'}$ is diagonal. The relation (9) then simply become:

$$\langle(\delta u_\ell)^n\rangle = a_n^{N_{\ell L}}\langle(\delta u_L)^n\rangle, \qquad (10)$$

where a_n is the nth diagonal component. This is exactly the relation found in [8].

2.3.2 Matrix property

The affine infinite divisibility shows that all the properties of scale variations of the moments are determined by the function $N_{\ell L}$, and by the elementary matrix A. This matrix has some interesting properties, which we explore now. First, by definition, A has has non-zero components only on and below its diagonal: it is a triangular matrix. So, we can write it as:

$$A = \begin{pmatrix} A_{00} & 0 & 0 & 0 & \dots \\ A_{10} & A_{01} & 0 & 0 & \dots \\ A_{20} & A_{11} & A_{02} & 0 & \dots \\ A_{30} & A_{21} & A_{12} & A_{03} & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}. \tag{11}$$

The normalization of the probability distribution F imposes $A_{00} = 1$. In homogeneous turbulence $\langle \delta u_\ell \rangle = 0$ at any scale. This imposes $A_{10} = 0$. Now, it is easy to find the expression of $A^{N_{\ell L}}$ by an iterative procedure. One finds:

$$A^{N_{\ell L}} = \begin{pmatrix} 1 & 0 & 0 & \dots \\ 0 & A_{01}^N & 0 & \dots \\ A_{20}\frac{1-A_{02}^N}{1-A_{02}} & A_{11}\frac{A_{01}^N - A_{02}^N}{A_{01}-A_{02}} & A_{02}^N & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}. \tag{12}$$

It is easy to show that the nth diagonal component of A^N is A_{0n}^N. As mentioned earlier, this is the "log-infinite divisibility" remnant.

2.4 Skewness

We can now check that this new formulation allows for skewness generation along the scales, as observed in turbulence. For this, we can decompose the propagator F into its odd and even parts, *via*:

$$S(F_{\ell\ell'}) = \frac{1}{2}\left(F_{\ell\ell'}(a,b) + F_{\ell\ell'}(a,-b)\right),$$
$$I(F_{\ell\ell'}) = \frac{1}{2}\left(F_{\ell\ell'}(a,b) - F_{\ell\ell'}(a,-b)\right). \tag{13}$$

In the matrix A, this decomposition corresponds a matrix with diagonal bands alternating with zero diagonal bands. For S, you keep the main diagonal of A^N, then the second next, the fourth next etc. For I, the main diagonal is zero, and you keep the next, the third next, the fifth next etc. Now, I can write a formula analog to (3) for the odd and the even part of the distribution function of the velocity increments which can be symbolically written:

$$S(P,\ell) = \int\int \left(S(P,\ell')S(F_{\ell\ell'}) + I(P,\ell')I(F_{\ell\ell'})\right),$$
$$I(P,\ell) = \int\int \left(S(P,\ell')I(F_{\ell\ell'}) + S(P,\ell')I(F_{\ell\ell'})\right), \tag{14}$$

where the two integrals are performed over a and b, and the general convolution between $S(P)$ and $S(F)$ etc is implicitly assumed. This formula shows that even starting from an initial distribution with no skewness $I(P) = 0$, $S(P) = P$ at scale L, one can generate skewness *via* the odd part of the propagator. If log-infinitely divisibility holds, like in Castaing formulation, this odd part is zero, and we find again the result that no skewness can be generated.

3 Discussion

We have proposed a generalization of Castaing formulation of log-infinitely divisibility for the velocity increments, which is potentially in better agreement with observations, because it allows for skewness generation along the scale. The new propagator that we introduced is now a joint probability distribution, and it would be very interesting to study experimentally is property, to see if the new formulation can help getting a better understanding about the scale variation of the statistics. The main problem is of course to get practical access to this propagator, but due to its shape, it is likely that it could be obtained *via* a wavelet analysis of the type already performed by Arneodo and its collaborators (see *e.g.* [6,7]).

The new formulation put emphasis on a new possible affine symmetry for the velocity increments, generalizing the multiplicative structure often assumed. It is interesting to stress that it lay in between two extreme statistical structure which have already been proposed for the velocity increments in the past: one is the additive structure, leading naturally to stable laws statistics, which has been proposed *e.g.* by Min *et al.* [10] in the case of 2D turbulence. This structure is related to the natural additive nature of velocity increments in homogeneous turbulence, since

$$\begin{aligned} \delta u_{2\ell} &= u(x+2\ell) - u(x) \\ &= u(x+2\ell) - u(x+\ell) + u(x+\ell) - u(x), \\ &= \delta u_\ell + \delta u_\ell. \end{aligned} \tag{15}$$

The second its the multiplicative structure, which was proposed in the case of 3D turbulence, to take into account the notion of cascade from scale to scale. These two structures are obviously complementary, since at short distances, the velocity increments are correlated and cannot be described with the additive structure. In the Langevin formulation of Friedrich *et al.* [1,2], the multiplicative and additive structure corresponds respectively to a multiplicative and an additive noise, which are correlated. We have actually shown elsewhere, that the multiplicative contribution can be viewed as stemming from the interaction with the large scale velocity field (the cascade would then comes stretching by large scales), while the additive contribution comes from pressure contribution [14]. It would be interesting to fit this findings into new phenomenological models of turbulence, to help building better closure models.

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